

## Super-Poincaré Covariant Two-Loop Superstring Amplitudes

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The super-Poincaré covariant formalism for the superstring is used to compute massless four-point two-loop amplitudes in ten-dimensional superspace. The computations are much simpler than in the RNS formalism and include both external bosons and fermions.

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## 1. Introduction

The computation of superstring multiloop amplitudes is important for testing duality conjectures and verifying finiteness. In the RNS formalism, the need to sum over spin structures leads to various complications in multiloop computations. For two-loop amplitudes with four massless external Neveu-Schwarz states, these complications were recently overcome in a series of papers by D'Hoker and Phong [1], and related papers by Iengo and Zhu et al [2]. Since the final expression for the four-point two-loop amplitude is remarkably simple, it is natural to ask if there is a more efficient computational method than the RNS formalism.

Five years ago, a new super-Poincaré covariant formalism for the superstring was introduced which uses pure spinors as worldsheet ghosts [3]. Last year, it was shown how to compute multiloop amplitudes using this formalism and various vanishing theorems were proven [4]. In this paper, this formalism will be used to compute massless four-point two-loop amplitudes in ten-dimensional superspace. The computation is much simpler than the RNS computations of [1][2] and automatically includes both external Neveu-Schwarz and external Ramond states.

As shown in [4], certain vanishing theorems related to finiteness are easily proven in the super-Poincaré covariant formalism by counting fermionic zero modes. To obtain the required number of fermionic zero modes for massless multiloop amplitudes, one needs at least four external states. And when there are precisely four external massless states, all fermionic worldsheet variables contribute only through their zero modes. This makes it relatively easy to evaluate four-point massless multiloop amplitudes and the four-point two-loop amplitude will be explicitly computed here. Higher-loop massless four-point amplitudes will hopefully be discussed in a later paper.<sup>2</sup>

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<sup>2</sup> Zhu has recently proposed in [5] a simple formula for four-point higher-loop Neveu-Schwarz amplitudes which has many features in common with the four-point two-loop Neveu-Schwarz expression. Unfortunately, his proposal needs to be modified since his definition of  $\Delta_g(z_i, z_j)$  in equation (12) of the first version of [5] is identically zero. This follows from the fact that  $\omega(P_I) = 0$  for  $I = 1$  to  $2g - 2$ , implying that  $\Delta_g(z_i, z_j)$  is proportional to  $\omega(z_i)\omega(z_j)$ , which vanishes after antisymmetrizing in  $i$  and  $j$ . Nevertheless, preliminary computations using the super-Poincaré covariant formalism suggest that the polarization and momentum dependence of higher-loop four-point amplitudes has the same structure as in the two-loop amplitude, in agreement with the proposal of Zhu.

## 2. Four-Point Two-Loop Computation

As discussed in [4], the four-point two-loop amplitude for the Type IIB superstring<sup>3</sup> is computed using the prescription:

$$\mathcal{A} = \int d^2\tau_1 d^2\tau_2 d^2\tau_3 \langle | \prod_{P=1}^3 \int d^2u_P \mu_P(u_P) \tilde{b}_{B_P}(u_P, z_P) \quad (2.1)$$

$$\prod_{P=4}^{20} Z_{B_P}(z_P) \prod_{R=1}^2 Z_J(v_R) \prod_{I=1}^{11} Y_{C_I}(y_I) \mid^2 \prod_{T=1}^4 \int d^2t_T U^{(T)}(t_T) \rangle,$$

where  $| \mid^2$  signifies the left-right product,  $\tau_P$  are the Teichmuller parameters associated to the Beltrami differentials  $\mu_P(u_P)$ ,  $\tilde{b}_{B_P}$  is the picture-raised  $b$  ghost,  $Z_{B_P}$  and  $Z_J$  are the picture-raising operators,  $Y_{C_I}$  are the picture-lowering operators, and  $U^{(T)}(t_T)$  are the dimension  $(1, 1)$  closed string vertex operators for the four external states. Using the notation of [4],  $\tilde{b}_{B_P}$  satisfies  $\{Q, \tilde{b}_{B_P}(u_P, z_P)\} = T(u_P)Z_{B_P}(z_P)$  where

$$Z_{B_P} = \frac{1}{2} B^{mn} (\lambda \gamma_{mn} d) \delta(B^{pq} N_{pq}), \quad Z_J = \lambda^\alpha d_\alpha \delta(J), \quad Y_{C_I} = C_{I\alpha} \theta^\alpha \delta(C_{I\beta} \lambda^\beta), \quad (2.2)$$

$N_{mn}$  and  $J$  are the Lorentz and ghost-number currents for the pure spinors, and  $B_P^{mn}$  and  $C_{I\alpha}$  are constant two-forms and spinors. As explained in [4], changing the choices for  $B_P^{mn}$  and  $C_{I\alpha}$  is a BRST-trivial operation which does not affect the scattering amplitude.

For massless external states,

$$U^{(T)} = e^{ik \cdot x} (\partial \theta^\alpha A_\alpha^{(T)}(\theta) + \Pi^m A_m^{(T)}(\theta) + d_\alpha W^{(T)\alpha}(\theta) + \frac{1}{2} N^{mn} \mathcal{F}_{mn}^{(T)}(\theta)) \quad (2.3)$$

$$(\bar{\partial} \bar{\theta}^\beta \bar{A}_\beta^{(T)}(\bar{\theta}) + \bar{\Pi}^p \bar{A}_p^{(T)}(\bar{\theta}) + \bar{d}_\beta \bar{W}^{(T)\beta}(\bar{\theta}) + \frac{1}{2} \bar{N}^{pq} \bar{\mathcal{F}}_{pq}^{(T)}(\bar{\theta}))$$

where the Type IIB supergravity vertex operator has been written as the left-right product of two super-Yang-Mills vertex operators. Using the convention  $D_\alpha = \frac{\partial}{\partial \theta^\alpha} + \frac{1}{2} k_m (\gamma^m \theta)_\alpha$ ,

$$A_\alpha(\theta) = \frac{1}{2} a_m (\gamma^m \theta)_\alpha - \frac{1}{3} (\xi \gamma_m \theta) (\gamma^m \theta)_\alpha + \dots \quad \text{and} \quad A_m(\theta) = a_m - (\xi \gamma^m \theta) + \dots \quad (2.4)$$

are the spinor and vector gauge superfields and

$$W^\alpha(\theta) = \xi^\alpha - \frac{1}{4} k_{[m} a_{n]} (\gamma^{mn} \theta)^\alpha + \dots \quad \text{and} \quad \mathcal{F}_{mn} = k_{[m} a_{n]} - k_{[m} (\xi \gamma_{n]} \theta) + \dots \quad (2.5)$$

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<sup>3</sup> This prescription is easily generalized to the Type IIA or heterotic superstring.

are the spinor and vector superfield-strengths of super-Yang-Mills, where  $(a_m, \xi^\alpha)$  are the on-shell (gluon, gluino) and ... involves  $a_m$  and  $\xi^\alpha$  with higher powers of  $k$  and  $\theta$ . The NS-NS, NS-R, R-NS, and R-R Type IIB supergravity vertex operators can be obtained from (2.3) by considering the terms proportional to  $a_m \bar{a}_n$ ,  $a_m \bar{\xi}^\beta$ ,  $\xi^\alpha \bar{a}_n$  and  $\xi^\alpha \bar{\xi}^\beta$  respectively.

At genus two, there are 16 fermionic zero modes for  $\theta^\alpha$  and 32 fermionic zero modes for  $d_\alpha$ . As in tree amplitudes, eleven of the  $\theta^\alpha$  zero modes can come from the  $Y_C$ 's and the remaining five  $\theta^\alpha$  zero modes will come from the external vertex operators. For the  $d_\alpha$  zero modes, nineteen  $d_\alpha$  zero modes can come from the seventeen  $Z_B$ 's and two  $Z_J$ 's, so thirteen  $d_\alpha$  zero modes must come from the three  $\tilde{b}_B$  ghosts and the four external vertex operators.

From the construction of  $\tilde{b}_B$  in [4], one finds that  $\tilde{b}_B$  contains terms with a maximum of four  $d$ 's, but does not contain any terms with three  $d$ 's. Since each vertex operator can contribute at most one  $d_\alpha$  zero mode, the only contribution from  $\tilde{b}_B$  comes from the terms with four  $d$ 's. One can show that all such terms are proportional to

$$H_B(z) = B_{mn} B^{qr} (d(z) \gamma^{mnp} d(z)) (d(z) \gamma_{pqr} d(z)) \delta'(B^{st} N_{st}(z)) \quad (2.6)$$

where  $\delta'(x)$  denotes  $\frac{\partial}{\partial x} \delta(x)$  and is defined to satisfy  $x \delta'(x) = -\delta(x)$ . Since each of the three  $\tilde{b}_B$  ghosts contains  $\delta'(BN)$  dependence, three of the vertex operators must contribute an  $N_{mn}$  zero mode to remove the derivative from the delta functions. Furthermore, the fourth vertex operator must contribute the last of the 32  $d_\alpha$  zero modes.

After performing the functional integration over the worldsheet nonzero modes, the amplitude prescription of (2.1) gives

$$\begin{aligned} \mathcal{A} = & \int d^2 \tau_1 d^2 \tau_2 d^2 \tau_3 \prod_{T=1}^4 \int d^2 t_T \frac{\exp(-\sum_{T,U=1}^4 k_T \cdot k_U G(t_T, t_U))}{(\det Im \Omega)^5} \quad (2.7) \\ & | \int [DC][DB][D\lambda][DN] \int d^{16} \theta d^{32} d \\ & \prod_{P=1}^3 \int d^2 u_P \mu_P(u_P) H_{B_P}(u_P) \prod_{P=4}^{20} Z_{B_P}(z_P) \prod_{R=1}^2 Z_J(v_R) \prod_{I=1}^{11} Y_{C_I}(y_I) \\ & \prod_{T=1}^4 (d_\alpha(t_T) W^{(T)\alpha}(\theta) + \frac{1}{2} N^{mn}(t_T) \mathcal{F}_{mn}^{(T)}(\theta)) |^2 \end{aligned}$$

where the factor of  $\frac{\exp(-\sum_{T,U=1}^4 k_T \cdot k_U G(t_T, t_U))}{(\det Im \Omega)^5}$  comes from the functional integration over the ten  $x$ 's,  $\Omega$  is the period matrix,  $G(t_T, t_U)$  is the usual scalar Green's function, and

$\int[DC][DB][D\lambda][DN]$  are measure factors for the pure spinor zero modes which are defined in [4]. The partition function vanishes in this formalism since the contribution from the ten  $x^m$  and 32  $(d_\alpha, \theta^\alpha)$  variables cancels the contribution from the 22 pure spinor variables.

To evaluate (2.7), first use the rules described in [4] to integrate over the zero modes of  $N_{mn}$  and  $d_\alpha$  and over the choices of  $B_{mn}$ . This produces the expression

$$\begin{aligned} \mathcal{A} = & \int d^2\tau_1 d^2\tau_2 d^2\tau_3 \prod_{T=1}^4 \int d^2t_T \frac{\exp(-\sum_{T,U=1}^4 k_T \cdot k_U G(t_T, t_U))}{(\det Im\Omega)^5} \\ & | \int [DC][D\lambda] \int d^{16}\theta \\ & \prod_{P=1}^3 \int d^2u_P \mu_P(u_P) \Delta(u_1, u_2) \Delta(u_2, u_3) \Delta(u_3, u_1) \prod_{I=1}^{11} Y_{C_I}(y_I) \lambda^\alpha \lambda^\beta \lambda^\gamma (\gamma^{mnpqr})_{\alpha\beta} \gamma_{\gamma\delta}^s \\ & ( F_{mn}^{(1)}(\theta) F_{pq}^{(2)}(\theta) F_{rs}^{(3)}(\theta) W^{(4)\delta}(\theta) \Delta(t_1, t_3) \Delta(t_2, t_4) + \text{permutations of } 1234 )^2 \end{aligned} \quad (2.8)$$

where  $\Delta(u, v) = \epsilon^{CD} \omega_C(u) \omega_D(v)$  and  $\omega_C(z)$  for  $C = 1, 2$  are the two holomorphic one-forms.

To derive (2.8) from (2.7), one uses that each  $H_{B_P}(u_P)$  has +2 conformal weight, has no poles on the surface, and has zeros when  $u_{P_1} = u_{P_2}$ . The unique such function is proportional to  $\Delta(u_1, u_2) \Delta(u_2, u_3) \Delta(u_3, u_1)$ . Similarly, the picture-raising operators have zero conformal weight and no poles, so they leave no contribution. And the external vertex operators have +1 conformal weight with no poles, so they contribute  $h^{CDEF} \omega_C(t_1) \omega_D(t_2) \omega_E(t_3) \omega_F(t_4)$  for some constant  $h^{CDEF}$ . Since the zero modes associated with  $\omega_1$  and  $\omega_2$  appear symmetrically,  $h^{CDEF}$  vanishes unless it has two 1 indices and two 2 indices, and is invariant under the exchange of the two 1 indices with the two 2 indices.

Moreover, Lorentz invariance implies that the three remaining  $\lambda$ 's must be contracted with the indices of the external superfields as

$$\lambda^\alpha \lambda^\beta \lambda^\gamma (\gamma^{mnpqr})_{\alpha\beta} \gamma_{\gamma\delta}^s F_{mn}^{(1)}(\theta) F_{pq}^{(2)}(\theta) F_{rs}^{(3)}(\theta) W^{(4)\delta}(\theta) \quad (2.9)$$

up to permutations of the external superfields.<sup>4</sup> And  $\lambda^\alpha \lambda^\beta \lambda^\gamma (\gamma^{mnpqr})_{\alpha\beta} \gamma_{\gamma\delta}^s (F_{mn}^{(1)} F_{pq}^{(2)} F_{rs}^{(3)} + F_{mn}^{(2)} F_{pq}^{(3)} F_{rs}^{(1)} + F_{mn}^{(3)} F_{pq}^{(1)} F_{rs}^{(2)}) = 0$  together with  $(\gamma^{mnpqr})_{\alpha\beta} (F_{mn}^{(1)} F_{pq}^{(2)} - F_{mn}^{(2)} F_{pq}^{(1)}) = 0$

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<sup>4</sup> This contraction can be shown to be unique by decomposing the (Wick-rotated)  $SO(10)$  representations into  $SU(5) \times U(1)$  representations. Under  $SU(5) \times U(1)$ ,  $W^\alpha$  decomposes into

implies that one can replace  $h^{CDEF}$  with  $\epsilon^{CE}\epsilon^{DF}$ . Note that by choosing the Teichmuller parameters to be the three elements of the period matrix  $\Omega_{CD}$ , one can write

$$\int d^2\tau_1 d^2\tau_2 d^2\tau_3 \left| \prod_{P=1}^3 \int d^2u_P \mu_P(u_P) \Delta(u_1, u_2) \Delta(u_2, u_3) \Delta(u_3, u_1) \right|^2 = \int d^2\Omega_{11} d^2\Omega_{12} d^2\Omega_{22}.$$

Finally, the integration over  $\int [DC][D\lambda] \int d^{16}\theta$  in (2.8) is easily performed using the rules of [4] to obtain

$$\begin{aligned} \mathcal{A} = & \int d^2\Omega_{11} d^2\Omega_{12} d^2\Omega_{22} \prod_{T=1}^4 \int d^2t_T \frac{\exp(-\sum_{T,U=1}^4 k_T \cdot k_U G(t_T, t_U))}{(\det Im\Omega)^5} \\ & |(\gamma^{mnpqr})_{\alpha\beta} \gamma_{\gamma\delta}^s \left( \int d^5\theta \right)^{\alpha\beta\gamma} \\ & (F_{mn}^{(1)}(\theta) F_{pq}^{(2)}(\theta) F_{rs}^{(3)}(\theta) W^{(4)\delta}(\theta) \Delta(t_1, t_3) \Delta(t_2, t_4) + \text{permutations of } 1234)|^2 \end{aligned} \quad (2.11)$$

where

$$\left( \int d^5\theta \right)^{\alpha\beta\gamma} = (T^{-1})_{\rho_1 \dots \rho_{11}}^{\alpha\beta\gamma} \epsilon^{\rho_1 \dots \rho_{16}} \left( \frac{\partial}{\partial \theta} \right)_{\rho_{12}} \dots \left( \frac{\partial}{\partial \theta} \right)_{\rho_{16}} \quad (2.12)$$

and  $(T^{-1})_{\rho_1 \dots \rho_{11}}^{\alpha\beta\gamma}$  is the  $\gamma$ -matrix traceless part of

$$\epsilon_{\rho_1 \dots \rho_{16}} (\gamma^m)^{\alpha\rho_{12}} (\gamma^n)^{\beta\rho_{13}} (\gamma^p)^{\gamma\rho_{14}} (\gamma_{mnp})^{\rho_{15}\rho_{16}}.$$

In other words,

$$(T^{-1})_{\rho_1 \dots \rho_{11}}^{\alpha\beta\gamma} = \epsilon_{\rho_1 \dots \rho_{16}} (\gamma^m)^{\alpha\rho_{12}} (\gamma^n)^{\beta\rho_{13}} (\gamma^p)^{\gamma\rho_{14}} (\gamma_{mnp})^{\rho_{15}\rho_{16}} + \gamma_m^{(\alpha\beta} E_{\rho_1 \dots \rho_{11}}^{\gamma)m} \quad (2.13)$$

where  $E_{\rho_1 \dots \rho_{11}}^{\gamma m}$  is defined such that  $\gamma_{\alpha\beta}^m (T^{-1})_{\rho_1 \dots \rho_{11}}^{\alpha\beta\gamma} = 0$ .

The four-point two-loop amplitude of (2.11) is remarkably simple. When all external states are chosen in the NS-NS sector, it should not be difficult to show agreement with the RNS formula of [1][2]. Work is currently in progress on extending these results to higher-loop four-point amplitudes.

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$[W_{\frac{5}{2}}^+, W_{\frac{1}{2}[ab]}, W_{-\frac{3}{2}}^a]$  and  $F_{mn}$  decomposes into  $[F_{+\frac{1}{2}}^{[ab]}, F_{0a}^b, F_{-2[ab]}]$  where  $a, b = 1$  to 5 and the subscript is the  $U(1)$  charge. Choosing  $\lambda^\alpha$  such that  $\lambda_{\frac{5}{2}}^+$  is the only nonzero component, one can easily verify that

$$(\lambda_{\frac{5}{2}}^+)^3 F_{-2[ab]}^{(1)} F_{-2[cd]}^{(2)} F_{-2[ef]}^{(3)} W_{-\frac{3}{2}}^{(4)f} \epsilon^{abcde} \quad (2.10)$$

is the unique  $SU(5) \times U(1)$  invariant term, which is written in  $SO(10)$ -invariant notation as (2.9).

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